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Acoustics of early universe. — Flat versus open universe models

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Abstract

A simple perturbation description unique for all signs of curvature, and based on the gauge-invariant formalisms is proposed to demonstrate that: (1) The density perturbations propagate in the flat radiation-dominated universe in exactly the same way as electromagnetic or gravitational waves propagate in the epoch of the matter domination. (2) In the open universe, sounds are dispersed by curvature. The space curvature defines the minimal frequency ω_{c} below which the propagation of perturbations is forbidden.

Gaussian acoustic fields are considered and the curvature imprint in the perturbations spectrum is discussed.

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1 Introduction

Discovering the wave nature of scalar perturbations in the early universe has a long history. Watchful reader of Harrison's classical paper [1] can guess the wave equations out of formulae given there (Section 5.5). Trigonometric or Bessel solutions together with $\omega\eta$-dependence characteristic for flat perturbed universes appear in both classical [2, 3] and gauge-invariant theories [4]. In the case of flat universe the gauge-specific wave equations are explicitly given by Sachs and Wolfe ([5] see theorem pp. 76–77). The comprehensive phonon description of perturbations in the flat radiation-filled universe, together with the attempt to quantize them, has been formulated by Lukash [6] and continued in its quantum aspect by others [7]. The wave character is confirmed [8] in the original Lifshitz-Khalatnikov [9] formalism. Acoustic motions of the baryon-electron system after recombination have been noticed by Yamamoto *at al.* [10]. Some parallels between scalar perturbation dynamics and gravitational waves can be found in [11, 12].

Controversies, however arise over the gravitational instability criteria, the gauge problems and the role of the space curvature.

(1) In the $\eta \to 0$ limit one can formally construct the growing and decaying solutions. Since these solutions are typically considered as the large scale approximation ($\omega \to 0$) the structure formation is expected in scales greater than the sound horizon. Consequently the Jeans criterion is understood as the dispersion relation dividing perturbations into two classes: acoustic waves and gravitationally bound structures. No dispersion relations like that can be inferred from the exact solutions [3]-[6], [8].

(2) As long as the results depend on the coordinate system (the gauge-specific solutions [5, 6, 8] differ one from another) their physical meaning is a subject of dispute. Acoustic field deserves complete gauge-invariant treatment.

(3) The problem of acoustic field does not seem to be solved properly in open universes, where most authors traditionally employ the flat space Fourier analysis, instead of Fourier expansions in the Lobachevski space.

In attempt to clear those points, we propose a simple perturbation description, which is unique for all signs of curvature, and based on the gauge-invariant perturbation formalisms (Sakai [3], Bardeen [4], Kodama and Sasaki [13], Lyth and Mukherjee [14], Padmanabhan [15], Brandenberger, Kahn and Press [16], Ellis, Bruni and Hwang [17], Olson [18], see also [19]).

Section 2 contains a brief recipe of how to reduce equations obtained in these theories to a single, second order partial differential equation (3). Differences between the formalisms occur to be of no importance here, and we obtain exactly the same propagation equation for all of them. We show how to transform this equation to the wave equation in its normal form.

We obtain a general, "profile-independent" solution for the flat universe (Section 3), without appealing to the Fourier transform. We demonstrate that the gauge-invariant density perturbation propagate in radiation-dominated universe in the same way as electromagnetic or gravitational waves propagate in the epoch of the matter domination. Eventually, we expand perturbations into planar waves, in order to discuss some basic features of the spectrum and the spectrum transfer function.

In section 4 we describe the sound propagation in open universes. We analyse the dispersive role of the curvature. The space curvature prevents perturbations of frequencies smaller than some critical $\omega_{\rm c}$ from propagating in space, and systematically reduces the group velocity for others, when ω goes down to $\omega_{\rm c}$.

Section 5 is devoted to Gaussian acoustic fields. We derive the spectrum transfer function in the form suitable to estimate the role of the space curvature in the microwave background.

2 Scalar perturbations in the early universe

In the universe filled with highly relativistic matter the energy momentum tensor is trace-free. The dynamics of the scale factor $a(\eta)$ expressed as a function of the conformal time η is governed by

$$T^{\mu}{}_{\mu} = -\frac{6}{a^3(\eta)}\left(a''(\eta) + K\,a(\eta)\right) = 0 \tag{1}$$

and yields

$$a(\eta) = \sqrt{\frac{\mathcal{M}}{3}}\frac{\sin\left(\sqrt{K}\eta\right)}{\sqrt{K}}. \tag{2}$$

We treat the curvature index K as continuous quantity and keep K explicitly in both the equations and solution, as far as possible. Traditional formulae can be recovered by setting $K = \pm 1$ or by the limit procedure $K \to 0$. Normalization $\sqrt{\mathcal{M}/3}$ recalls the constant of motion $\mathcal{M} = \rho(\eta)a^4(\eta)$.

The perturbation equation expressed in the orthogonal gauge[1], [18, 19] and parameterized by the conformal time[2] η takes the canonical form (free of first derivatives)

$$\frac{\partial^2}{\partial\eta^2}X(\eta,\mathbf{x}) - \frac{2\mathcal{M}}{3a^2(\eta)}X(\eta,\mathbf{x}) - \frac{1}{3}\,{}^{(3)}\Delta X(\eta,\mathbf{x}) = 0 \tag{3}$$

where ${}^{(3)}\Delta$ denotes the Laplace-Beltrami operator acting on orthogonal hypersurfaces.

Equation (3) can be easily derived from the Raychaudhuri and the continuity equations (see the procedure in [14] or [15]). It also can be recovered from the Sakai equation [3] (formula (5.1), $K \to X$), the equation for density perturbations in orthogonal gauge (Bardeen's [4] formula (4.9), $\rho_{\mathrm{m}} \to X$, Kodama and Sasaki [13] chap. IV, formula (1.5), $\mathit{\Delta} \to X$, Lyth and Mukherjee [14] formulae (16–17), $\delta \to X$, Padmanabhan [15] Eq. (4.88), $\delta \to X$), the equation for gauge invariant metric potentials (Brandenberger, Kahn and Press [16] formula (3.35), $\mathit{\Phi}_H/\rho a^2 \to X$), the equation for gauge invariant density gradients (Ellis, Bruni and Hwang [17] formula (38), $\mathcal{D} \to X$) or Laplacians (Olson [18] formulae (8–9), as well as its extension to open universes [19] formula (22)) after transforming these equations to conformal time (if parameterized differently) and employing the Helmholtz equation to restore partial form of the perturbation equation. Suitable changes of the variable names as indicated above (*original* $\to X$) are necessary.

We introduce a new perturbation variable $\widehat{X}$

$$\widehat{X}(\eta,\mathbf{x}) = \frac{1}{a(\eta)}\frac{\partial}{\partial\eta}(a(\eta)X(\eta,\mathbf{x})). \tag{4}$$

While $X(\eta,\mathbf{x})$ satisfies (3) and $a(\eta)$ is given by (2) the perturbation variable $\widehat{X}$ obey the wave equation in its normal form

$$\frac{\partial^2}{\partial\eta^2}\widehat{X}(\eta,\mathbf{x}) - \frac{1}{3}\,{}^{(3)}\Delta\widehat{X}(\eta,\mathbf{x}) = 0. \tag{5}$$

The time derivatives of the gauge-invariant inhomogeneity measures also form gauge-invariant variables. In this sense variable $\widehat{X}$ is equally good as X.

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[1] or the gauge-invariant differential measures of inhomogeneity [17]

[2] In the orthogonal gauge the conformal time is defined as the integral $\eta = \int \frac{1}{a(t)}\mathrm{d}t$, where time t means the orthogonal time — the time parameter constant on orthogonal hypersurfaces.

However, time derivatives may be difficult to observe at the last scattering surface, and hardly represent physically meaningful aspects of the cosmic structure. The equation (5) plays only an auxiliary role, nevertheless is very useful. Formally, it describes the wave (massless field) propagating in the static space-time of constant space-curvature. The specific case of positive curvature (Einstein static universe) has been considered [20] in the context of quantum field theory on curved background. In our case spaces of zero or negative curvature are of particular importance. We will discuss both cases individually.

3 Sound waves on the flat background

When the space curvature vanishes the equation (3) reads as

$$\frac{\partial^2}{\partial\eta^2}X(\eta,\mathbf{x}) - \frac{2}{\eta^2}X(\eta,\mathbf{x}) - \frac{1}{3}\,{}^{(3)}\Delta X(\eta,\mathbf{x}) = 0 \tag{6}$$

and is essentially the same as the propagation equation for gravitational [11, 12] or electromagnetic[3] waves in the dust-filled universe

$$\frac{\partial^2}{\partial\eta^2}X(\eta,\mathbf{x}) - \frac{2}{\eta^2}X(\eta,\mathbf{x}) - {}^{(3)}\Delta X(\eta,\mathbf{x}) = 0. \tag{7}$$

The only differences are that gravitational and electromagnetic waves are expressed by the tensor $h_{\mu\nu}$ and vector A_μ respectively, and they propagate with the speed of light ($c = 1$), while the solutions to equation (6), represent scalar waves travelling with the velocity $v = 1/\sqrt{3}$.

Now, the Laplacian ${}^{(3)}\Delta$ operates in Euclidean space. Equation (5) when expressed in Cartesian coordinates $\{\mathbf{x}\}$ is solved by an arbitrary function $\widehat{X} = \widehat{X}(\mathbf{n}\cdot\mathbf{x} - v\eta)$ [22] with $v = 1/\sqrt{3}$ and $\mathbf{n}\cdot\mathbf{n} = 1$. However, to keep the linear approximation valid we require that $\widehat{X}(\mathbf{n}\cdot\mathbf{x} - v\eta)$ to be limited throughout the space[4].

Knowing the general solutions of the equation (5) we can return to observables X. We look for the general form of $X(\eta,\mathbf{x})$ among the solutions of

[3] See formula (5.2.6) in [21] after substituting $g = \eta^6$.

[4] $\forall\eta\,\exists\epsilon \in \mathbb{R}$: $\forall\mathbf{x} \quad -\epsilon < \widehat{X}(\mathbf{n}\cdot\mathbf{x} - v\eta) < \epsilon$.

the equation (4)

$$X(\eta,\mathbf{x}) = \frac{1}{\eta}\left(\int_0^{\eta}\eta'\,\widehat{X}(\mathbf{n}\cdot\mathbf{x}-v\eta')\mathrm{d}\eta' + F(\mathbf{x})\right). \tag{8}$$

On the strength of (6) $F(\mathbf{x})$ is harmonic ${}^{(3)}\Delta F(\mathbf{x}) = 0$ and must be constant if limited throughout the space of constant curvature [23]. With no loss of generality[5], we put $F(\mathbf{x}) = 0$. Eventually, the general, spatially limited solution to the equation (6) is expressed by the integral

$$X(\eta,\mathbf{x}) = \frac{1}{\eta}\int_0^{\eta}\eta'\,\widehat{X}(\mathbf{n}\cdot\mathbf{x}-v\eta')\mathrm{d}\eta' \tag{9}$$

of an arbitrary, but also spatially limited function $\widehat{X}(\mathbf{n}\cdot\mathbf{x}-v\eta)$. The solution describes a wave having the time-dependent profile and travelling with the constant velocity $v = 1/\sqrt{3}$.

This can be easily confirmed by the Fourier expansion analysis. Indeed, for any real function $\widehat{X}(\mathbf{n}\cdot\mathbf{x}-v\eta)$ expressed as

$$\widehat{X}(\eta,\mathbf{x}) = \int(\mathtt{A}_k\mathtt{u}_k(\eta,\mathbf{x}) + \mathtt{A}^*_k\mathtt{u}^*_k(\eta,\mathbf{x}))\mathrm{d}\mathbf{k} \tag{10}$$

with

$$\mathtt{u}_k(\eta,\mathbf{x}) = \frac{1}{\sqrt{2\omega}}\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{x}-\omega\eta)} \tag{11}$$

corresponds $X(\eta,\mathbf{x})$

$$X(\eta,\mathbf{x}) = \int(\mathcal{A}_k\,u_k(\eta,\mathbf{x}) + \mathcal{A}^*_k\,u^*_k(\eta,\mathbf{x}))\mathrm{d}\mathbf{k} \tag{12}$$

expanded into modes $u_k(\eta,\mathbf{x})$

$$u_k(\eta,\mathbf{x}) = \mu_\omega(\eta)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} = \frac{1}{\sqrt{2\omega}}\left(1+\frac{1}{i\omega\eta}\right)\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{x}-\omega\eta)} \tag{13}$$

The frequency ω obeys the dispersion relation $\omega^2 = k^2/3$, the Fourier coefficient $\mathcal{A}_k = -\frac{1}{i\omega}\mathtt{A}_k$ is an arbitrary complex function of the wave number

[5] The freedom to choose this constant is not different from ambiguity in the indefinite integral in (8). Appropriate integration constants are traditionally tuned to give the perturbation spatial average equal to zero.

k, and u_k is obtained from (9) after substituting $\widehat{X} = \mathtt{u}_k$. Modes u_k like $\mathtt{u}_k$ form an orthonormal base in the function space with the Klein-Gordon scalar product [20]. Both $\mathtt{u}_k$ and u_k form travelling waves, but only $\mathtt{u}_k$ have their absolute value constant in time. Therefore, the generic perturbation $X(\eta, \mathbf{x})$ is composed of plane waves u_k of decaying amplitude, which perfectly agrees with Sachs and Wolfe results ([5] pp. 76–77) obtained in alternative perturbation approach. Waves move with constant velocity $v = 1/\sqrt{3}$ independently of their length-scale. Short-scale and long-scale perturbations do not form different classes of solutions.

In the theory appealing to stochastic processes the initial perturbation is given at random at the end of the quantum epoch $\eta_i > 0$, and develops gravitationally according to (6) in the interval $\eta > \eta_i$. Therefore, solution's singularity at $\eta = 0$ is purely mathematical fact with no physical consequences.

4 Sound waves in the curved space

While decomposing perturbations into Fourier series in flat or opened universes we should respect some specific effects caused by the curvature. This particularly refers to open universes (the Lobachevski space), where the orthogonal expansions exist only for a class of perturbations with sufficiently short-scale autocorrelation [24]. To expand the others one needs supplementary series (supercurvature modes [25, 26, 27, 28]) of non-orthogonal[6] solutions to the Helmholtz equation, which are numbered by imaginary wave numbers $k \in [-i, i]$.

Let us adopt spherical coordinates $\{r, \theta, \phi\}$ as more appropriate for curved maximally symmetric spaces. In the same manner like in the scalar field theories [29] the density perturbation expands as

$$X(\eta, r, \theta, \phi) = \sum_{lm} \int (A_{klm}\, u_{klm}(\eta, r, \theta, \phi) + A^*_{klm}\, u^*_{klm}(\eta, r, \theta, \phi)) \mathrm{d}k \qquad (14)$$

where modes $u_{klm}(\eta, r, \theta, \phi) = \mu_{\omega,K}(\eta) Y_{klm}(r, \theta, \phi)$ are expressed by hyperspherical harmonics $Y_{klm}(r, \theta, \phi)$ and time-dependent amplitude $\mu_{\omega,K}(\eta)$ ful-

[6] In the sense of the Klein-Gordon scalar product.

filling the time-equation (obtained by separation from (3)):

$$\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\mu_{\omega,K}(\eta) + \left(\frac{k^2 - K}{3} - \frac{2K}{\sin^2(\sqrt{K}\eta)}\right)\mu_{\omega,K}(\eta) = 0. \tag{15}$$

We find solutions to (15) in the exact form as

$$\mu_{\omega,K}(\eta) = \frac{1}{\sqrt{2}}\sqrt{\frac{\omega}{\omega^2 - K}}\left(1 + \sqrt{K}\,\frac{\cot(\sqrt{K}\eta)}{i\omega}\right)\mathrm{e}^{-i\omega\eta}. \tag{16}$$

and their complex conjugates. Solutions $\mu_{\omega,K}(\eta)$ approach $\mu_\omega(\eta)$ (eq. 13) in the $K \to 0$ limit. The frequency ω and the wave number are related to each other by the dispersion relation

$$\omega(k) = \frac{\sqrt{k^2 - K}}{\sqrt{3}}. \tag{17}$$

which can be obtained by simple substitution of (16) into (15) and perfectly agrees with the dispersion relation obtained for the variable Y on the strength of equation (5) (compare [20] chapter 5.2).

Functions $Y_{klm}(r, \theta, \phi)$ solve the Helmholtz equation [20]

$$^{(3)}\Delta Y_{klm}(r, \theta, \phi) = -(k^2 - K)Y_{klm}(r, \vartheta, \phi) \tag{18}$$

and can be split into the radial part Π_{kl} and the two dimensional spherical functions $Y_{lm}(\vartheta, \varphi)$

$$Y_{klm}(\chi, \vartheta, \varphi) = \Pi_{kl}(\chi)Y_{lm}(\vartheta, \varphi) \tag{19}$$

Solutions to radial equation

$$\frac{\partial^2}{\partial\chi^2}\Pi_{kl}(\chi) + 2\coth\chi\frac{\partial}{\partial\chi}\Pi_{kl}(\chi) - \left(\lambda + \frac{l(l+1)}{\sinh^2\chi}\right)\Pi_{kl}(\chi) = 0 \tag{20}$$

are given by

$$\begin{aligned}
\Pi_{kl} &= N_{kl}\widetilde{\Pi}_{kl} \\
N_{kl} &= \sqrt{\frac{2}{\pi}}k^2\left[\prod_{n=0}^{l}(n^2 + k^2)\right]^{-1/2} \\
\widetilde{\Pi}_{kl} &= (k^2\sinh\chi)^l\left(\frac{-1}{k\sinh\chi}\frac{\mathrm{d}}{\mathrm{d}(k\chi)}\right)^{l+1}\cos(k\chi)
\end{aligned}$$

The lowest multipole solutions

$$\widetilde{\Pi}_{k0} = \frac{1}{\sinh\chi}\left(\frac{\sin k\chi}{k}\right)$$
$$\widetilde{\Pi}_{k1} = \frac{1}{\sinh\chi}\left(-\cos k\chi + \coth\chi\frac{\sin k\chi}{k}\right)$$
$$\widetilde{\Pi}_{k2} = \frac{1}{\sinh\chi}\left(-3\coth\chi\cos k\chi + (3\coth^2\chi - k^2 - 1)\frac{\sin k\chi}{k}\right)$$

are enough to demonstrate properties of both two series of hyperspherical harmonics. For real wave numbers ($k^2 > 0$) the Π_{k1} (consequently also $Y_{klm}(r, \theta, \phi)$) functions oscillate in space. They form an orthonormal basis in the sense of the scalar product $(f_1|f_2) = \int f_1 f_2^* \sqrt{g}\, \mathrm{d}^3x$. As proved by Gelfand and Naimark they are complete to expand square integrable functions in the Lobachevski space [30, 31]. For imaginary wave numbers contained in the interval $-1 \leq k^2 < 0$, the Π_{k1} (and $Y_{klm}(r, \theta, \phi)$) functions build supplementary series. These functions are regular, limited but strictly positive throughout space, so they are not orthogonal[7]. The supplementary series is redundant for expansion of square integrable functions. Nevertheless, this series is necessary to expand weakly homogeneous stochastic processes in the Lobachevski space [25, 26].

In this way in the open universe one obtains two types of hyperspherical harmonics $Y_{klm}(r, \theta, \phi)$ and consequently two types of modes $u_{klm}(\eta, r, \theta, \phi)$. Modes $u_{klm}(\eta, r, \theta, \phi)$ with real k are orthogonal by means of Klein-Gordon scalar product [29] and expand waves of square integrable profile. Modes $u_{klm}(\eta, r, \theta, \phi)$ with $-1 \leq k^2 < 0$ form "waves of infinite length-scale". Both types of modes may contribute to the spectrum of randomly (or quantum) originated inhomogeneities [27, 28].

[7]Modes with $k = \pm i$ are constant throughout space. One can subtract them by suitable changes in the background metric. Other modes with $k \in (-i, i)$, although positive everywhere, decrease with distance strong enough to keep zero mean value. For instance, for the spherically symetric perturbation with the density excess given by $\rho(\chi) = \Pi_{\frac{i}{2}0}(\chi) = 2\,\mathrm{csch}(\chi)\sinh\left(\frac{\chi}{2}\right)$ the mass-to-volume ratio $\overline{\rho}(r) = \frac{4\pi\int_0^r \rho(\chi)\sinh^2(\chi)\mathrm{d}\chi}{4\pi\int_0^r \sinh^2(\chi)\mathrm{d}\chi}$ tends to zero with volume tending to infinity: $\overline{\rho}(r) \longrightarrow \frac{\rho(r)\sinh^2(r)}{\sinh^2(r)} \longrightarrow 2\,\mathrm{csch}(r)\sinh\left(\frac{r}{2}\right) \longrightarrow 0$. No redefinition of the background can absorb perturbations like that.

The density perturbations propagate in the open universe in different manner than the scalar fields or gravitational waves do. Acoustic waves of different length-scales propagate with different velocities. Indeed from relation (17) we can infer both the phase and group velocity of sound in the form

$$v_{\rm f}(k) = \frac{\omega(k)}{k} = \frac{\sqrt{1+k^2}}{\sqrt{3}k} \tag{21}$$

and

$$v_{\rm g}(k) = \frac{\partial}{\partial k}\omega(k) = \frac{k}{\sqrt{3}\sqrt{1+k^2}}. \tag{22}$$

The group velocity decreases with the wave number k, to vanish completely at the $k \to 0$ limit. The condition $k = 0$ determines the critical frequency $\omega(0) = 1/\sqrt{3}$, below which the wave propagation is forbidden. Therefore, the acoustic travelling waves are composed of the principal series modes. The supplementary series build 'global' standing waves of supercurvature scale.

5 Gaussian acoustic field

The generic acoustic field described by (5) is composed of waves travelling in different directions. Clearly, this property also refers to solutions of the equation (3). The mechanism, which creates initial small perturbations is expected to be of probabilistic nature (thermodynamic or quantum fluctuations), therefore the evolution of linear structure is usually expressed in the language of stochastic processes. The homogeneity of this stochastic process reflects the universe homogeneity. Weakly homogeneous processes have their Fourier expansions

$$\widehat{X}(\eta,\mathbf{x}) = \int \left(\mathtt{A}_k \mathtt{u}_k(\eta,\mathbf{x}) + \mathtt{A}_k^* \mathtt{u}_k^*(\eta,\mathbf{x})\right) \mathrm{d}\mathbf{k} \tag{23}$$

and consequently

$$X(\eta,\mathbf{x}) = \int \left(\mathcal{A}_k u_k(\eta,\mathbf{x}) + \mathcal{A}_k^* u_k^*(\eta,\mathbf{x})\right) \mathrm{d}\mathbf{k} \tag{24}$$

where the coefficient $\mathcal{A}_k$ are random variables of k and the integral has a stochastic sense [32]. In a *generic Gaussian field*[8] the expectation values for

[8] Allen and collaborators [33] distinguish between *stationary* random fields fulfilling (25-26) and the *squeezed* random fields, which violates (26). Stationary processes have

$\mathcal{A}_k$ fulfill

$$E[\mathcal{A}_k \mathcal{A}^*_{k'}] = \mathcal{P}_k \delta_{kk'}, \tag{25}$$
$$E[\mathcal{A}_k \mathcal{A}_{k'}] = 0. \tag{26}$$

$\mathcal{P}_k$ is defined as the field spectrum. The first relation guarantees the statistical independence of waves of different wave numbers, the second — says that no particular phase is preferred. Waves moving in different directions are statistically independent.

The temperature fluctuations at the last scattering surface draw our attention to spatial correlations of $\delta\rho/\rho$ measured at the instant $\eta = \eta_{\mathrm{r}}$. In the flat universe the two-point spatial autocorrelation $R(h)$ [24] of the field X given by (16) can be expressed as:

$$R(h) = \frac{1}{4\pi} \int E[X(\mathbf{x}, \eta) X[\mathbf{x} + \mathbf{h}, \eta)] \delta(\mathbf{h}\cdot\mathbf{h} - 1)\, \mathrm{d}\mathbf{h} \tag{27}$$
$$= \frac{1}{4\pi} \int 2 u_k u^*_k \mathcal{P}_k \exp(i\mathbf{k}\cdot\mathbf{h}) \delta(\mathbf{h}\cdot\mathbf{h} - 1)\, \mathrm{d}\mathbf{k}\, \mathrm{d}\mathbf{h} \tag{28}$$
$$= \int_0^\infty 4\pi k^2 j_0(hk) 2\mu_\omega \mu^*_\omega \mathcal{P}_k\, \mathrm{d}k \tag{29}$$
$$= \int_0^\infty 4\pi k^2 \frac{\sin(hk)}{hk} p_k(\eta)\, \mathrm{d}k \tag{30}$$

where j_0 is the spherical Bessel function, and p_k stands for the space spectrum of the density perturbation at a given moment η. Following Peebles [34] we define the transfer function $T_\omega(\eta) = 2\mu_\omega \mu^*_\omega$, which converts the time-invariant field spectrum $\mathcal{P}_k$ into the space spectrum $p_k(\eta)$.

$$p_k(\eta) = T_\omega(\eta) \mathcal{P}_k = 2\mu_\omega \mu^*_\omega \mathcal{P}_k. \tag{31}$$

The formula for the space spectrum $p_k(\eta)$ splits into two factors: 1) $T(\eta)$ — describing the role of gravity, and 2) $\mathcal{P}_k$ coming from other interactions and rendering their probabilistic nature prior or during radiational era. We do not discus any specific form of $\mathcal{A}_k$ or $\mathcal{P}_k$ in this paper. We assume, however, that $\mathcal{A}_k$ enables one to construct small perturbations, and in particular does not cause divergences in Fourier integrals. Employing (13) one easily founds

$$T_\omega(\eta) = \frac{1}{\omega} \left(1 + \frac{1}{(\omega\eta)^2} \right). \tag{32}$$

their precise meaning in the framework of the stochastic theory not equivalent to (25-26). We prefer to name these random fields *generic* to stress analogy to generic classical fields.

As seen from (32) the evolution of each mode depends on the product $(\omega\eta)^2$. The contribution from modes much larger than the horizon scale $\omega\eta \ll 1$ strongly decreases with time, while the perturbation well inside the horizon $\omega\eta \gg 1$ keep constant amplitude. This property confirms the stability of Robertson-Walker symmetry against the generic (both classical and stochastic) large-scale density perturbations.

In the same manner one can express random acoustic fields in the open universe

$$X(\eta, r, \theta, \phi) = \sum_{lm} \int (A_{klm}\, u_{klm}(\eta, r, \theta, \phi) + A^*_{klm}\, u^*_{klm}(\eta, r, \theta, \phi))\mathrm{d}k. \tag{33}$$

According to Yaglom's theorem, for weakly homogeneous processes integration runs over both series principal and supplementary. The autocorrelation function now reads [26]:

$$R(h) = \int\limits_{R_+ \cup [0,i]} 4\pi k^2 \frac{\sin(hk)}{k \sin(h)} p_k(\eta)\mathrm{d}k \tag{34}$$

and the transfer function determined from formula (16) and (31) takes the form

$$T_\omega(\eta) = \frac{1}{\omega}\left(1 + \frac{1}{\omega^2 - K}\frac{K}{\sinh^2(\sqrt{K}\eta)}\right) \tag{35}$$

We rewrite $T_\omega(\eta)$ as a function of the energy density

$$T_\omega(\rho) = \frac{1}{\omega}\left(1 + \frac{\sqrt{\mathcal{M}\rho}}{3(\omega^2 - K)}\right) \tag{36}$$

to demonstrate that the curvature modifies substantially the space spectrum p_k only in the low frequency limit (supercurvature modes). Therefore, one may expect to find the curvature signature mostly in low multipoles (dipole, quadruple) [27]. To extract this geometrical effect the knowledge of the field spectrum $\mathcal{P}_k$ is indispensable.

Expansions (12) are typically employed in the gravitational waves theory [12], and in the scalar field theory [29], while the density perturbations theories traditionally solve, basically the same, propagation equation in terms of the Bessel $J_{3/2}$ and Neumann $N_{3/2}$ functions. The $J_{3/2}$ and $N_{3/2}$ are identified with "growing" and "decaying" modes respectively, according to their limit

behaviour at $\eta = 0$. Since the transition from $\{u, u^*\}$ basis to $\{J_{3/2}, N_{3/2}\}$ is the unitary transformation, both representations are equivalent. When the "decaying" mode is rejected [35] (a "standard practice" in cosmology — see comments in [36]) the unitarity is broken down and solution space is truncated to the space of standing waves. Then the acoustic field is in a highly "squeezed state" and consequently the characteristic peaks in the transfer function appears [35].

6 Remarks on scales and observables

There is a substantial difference between the dispersion on curvature, we described above, and "the curvature imprint" in CMBR spectrum, anticipated by the acoustic peaks hypothesis. The last hypothesis claims that the early perturbed universe was overdominated by stationary waves [35]. One may justify this assumption by appealing to squeezing phenomena at the transition from deSitter phase to the radiation dominated epoch. (In a transitions like that the large-scale stationary gravitational waves are generated (see [37]). The same refers to massless scalar field.) The dominance of standing waves with specifically correlated phases should exhibit a series of peaks in the CMBR spectrum. Positions of these peaks are sensitive to details of the universe dynamics (Ω, Λ, etc.).

On the contrary, the dispersion effect described in the section 4 has strictly geometrical character. It comes directly from the wave equation in the radiational epoch and does not depend on universe's past (evolution prior to radiational era is irrelevant here). The radiation-filled universe become "opaque" to sound waves greater than the curvature radius, whatever is the sound origin. In particular, no additional mechanism preferring standing waves at the beginning of the radiational era is needed. On that account the dispersion might form a reliable curvature tracer, provided it is observable at all, i.e.

1) the space scales of supercurvature perturbations must be "small enough" to fit well in the observable part of our universe, and

2) observational data should be complete enough to distinguish between standing and travelling waves at the last scattering.

Answer to the first question is relatively easy and it was already formulated in the literature in terms of multipole decomposition [27]. We repeat

the same result below by use of simple geometrical consideration. Let us assume we live in the open ($\Omega = 0.2$) universe which is presently dominated by matter ($p = 0$) . We see the last scattering surface at some η_r with the redshift $z_r = 1000$. For sake of simplicity (following [37]) we assume an instant transition from radiational epoch ($p = \rho/3$) to the galactic era ($p = 0$), which occur just at the last scattering moment $\eta = \eta_r$. In such a universe model the scale factor evolves as

$$a(\eta) = \sqrt{\frac{\mathcal{M}}{3}} \frac{\sinh^2\left(\frac{\eta+\eta_r}{2}\right)}{\sinh(\eta_r)}$$

and the radius of the visible universe χ_r can be easily expressed as a function of redshift z_r and the cosmological parameter Ω

$$\chi(\Omega, z_r) = 2\,\mathrm{arccoth}\left(\frac{1}{\sqrt{1-\Omega}}\right) - 2\,\mathrm{arcsinh}\left(\sqrt{\frac{1-\Omega}{\Omega(1+z_r)}}\right)$$

Setting typical values $\Omega = 0.2$ and $z_r = 1000$ one obtains $\chi_r = 2.76$. The equator plane "draws" on the last scattering surface a circle of the perimeter $l = 2\pi \sinh(\chi_r) = 49.5$, consequently, the curvature radius (in this units equal to one) takes $\alpha = 360/l = 7.28$ degrees on the sky. Let us consider now a spherically symmetric density perturbation (on the last scattering surface) described by the $k = 0$ hyperspherical function $Y_{000} = \chi\,\mathrm{csch}(\chi)$. The Y_{0lm} functions, with arbitrary l and m, form a boundary between the principal and supplementary series, and can be understood as the "shortest" supercurvature modes. Since these functions are positive everywhere, we express the perturbation length-scale as the half-magnitude width $l_{1/2}$. For spherically symmetric mode Y_{000} it is roughly $l_{1/2} = 2.2$. This corresponds to 32° on the sky. Equator intersects about 10 patches of that size. The supercurvature perturbations contribute mainly to the lowest multipoles in the CMBR spectrum (roughly $l < 5$), but the visible part of the universe is large enough to produce the curvature effects. Another problem is how to determine from the CMBR data, which waves are stationary and which of them are travelling ones. The oscillation time scale for waves close to the critical frequency $\omega_c = \sqrt{1/3}$ is of the order of the universe age, thus no direct observation of their temporal behaviour can be done. The same refers to gravitational squeezed waves (see [33]). On the other hand, the information we need is "hidden" on the last scattering surface, and the "only

task" is to read it properly. For standing perturbations the density and the velocity fields are strongly correlated [38]. Density extrema coincide with the expansion extrema throughout the entire space[9]. This means that even in random acoustic fields the density and the velocity perturbations loose their statistical independence in scales comparable with the space curvature and consequently would gradually correlate on the sky when the angle scales outgrow ten degrees. The key to solve this problem is to find a second independent observable, which would help us to separate the potential and Doppler contributions to the CMBR temperature fluctuations. We can hardly propose a definite candidate at the present stage of observations, but both the polarization measurements and the large scale flows analysis seems to be steps in the right direction.

7 Summary

The gauge-invariant analysis confirms that density perturbations in the radiation dominated universe form a field of acoustic waves. In the flat universe the density perturbations of all length-scales move with the same sound speed $v = 1/\sqrt{3}$. Short and long perturbations do not form different classes of solutions. Perturbations' velocity is independent of the wave number, and in particular is the same for subhorizon and superhorizon inhomogeneities. Although the gauge-invariant theory confirms the fundamental properties of acoustic field, which have been known from gauge-specific descriptions [5, 6, 8], the propagation equations are not the same. An outstanding property of the gauge-invariant description is that propagation equation for sound in radiational era are identical with the propagation equations for gravitational or electromagnetic waves in the matter dominated universe.

In the open universe perturbations evolve in a more complex manner. The negative space curvature causes the dispersion of acoustic waves. The universe geometry determines the minimal frequency for traveling acoustic waves in the similar way as the geometry of the wave conductor determines the minimal frequency for waves propagating inside. The critical frequency is related solely to the space curvature (not to the Jeans length-scale). Below this frequency perturbations form standing waves of supercurvature scale.

[9]Considerably more complex coincidences may be expected when multifluid models are taken into account. These cases need independent investigations.

In the radiation dominated universe the distinction between travelling and standing acoustic waves strictly coincides with the division into subcurvature and supercurvature inhomogeneities. Supercurvature standing waves are generic solutions.

As commonly expected, the spectrum transfer function depends on the universe geometry, but differences are essential only in the large scale limit. In the subcurvature regime *generic Gaussian acoustic field* evolve like the acoustic field in the flat universe. Significant curvature effects may appear in supercurvature scales — lowest multipoles in the MBR temperature map.

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